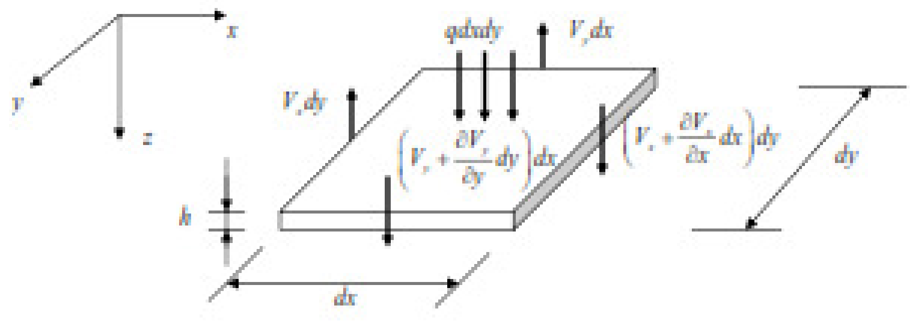
Unit IV – YIELD LINE THEORY

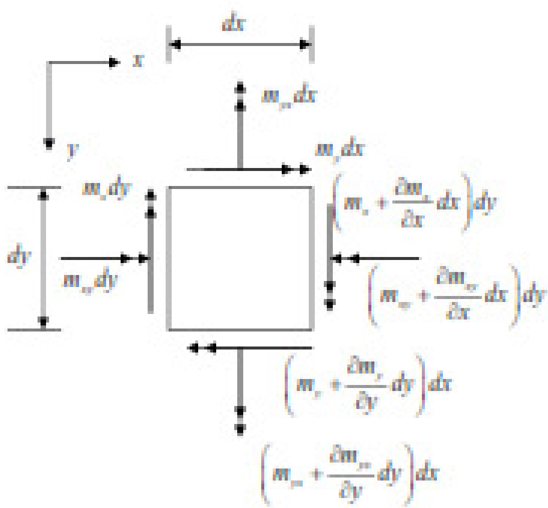
Application of virtual work method to square, rectangular, circular and triangular slabs.

Yield Line Theory for Slabs

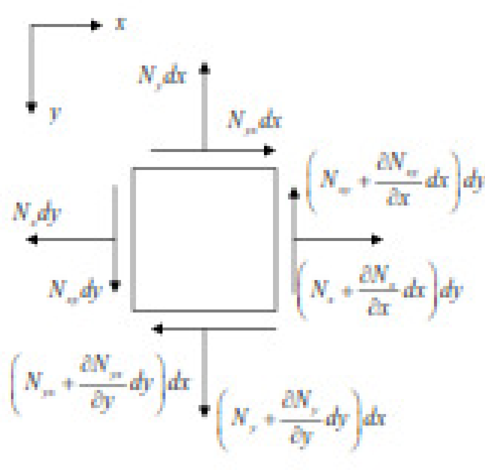
- Loads and load effects



Surface and shear forces



Moments



Membrane forces

- Load effects to be solved: $V_x, V_y, m_x, m_y, m_{xy}, m_{yx}, N_x, N_y, N_{xy}, N_{yx}$
  - → Ten unknowns and six equations
  - → Indeterminate problem: We need to include stress-strain relation for complete elastic solution.
- The relative importance of the load effects is related to the thickness of the slab. Most reinforced and prestressed concrete floor slabs fall within "medium-thick" class, i.e., plates are
  - thin enough that shear deformations are small, and
  - thick enough that in-plane or membrane forces are small.

Analysis methods:

- Elastic theory
- Elastic-plastic analysis
  – Finite element analysis (FEA)
- Approximate methods of analysis
- Limit analysis – **Yield Line Theory**
  – Lower & upper bound analysis

Elastic theory

- Lagrange's fourth-order PDE governing equation of isotropic plates loaded normal to their plane:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

where

w = deflection of plate in direction of loading at point (x, y).

q = loading imposed on plate per unit area, $q \approx f(x, y)$

D = flexural rigidity of plate, $D = \frac{Eh^3}{12(1-\mu^2)}$

E = Young's modulus

h = plate thickness

μ = Poisson's ratio.

- Navier's solution of Lagrange's equation using doubly infinite Fourier series:

  $$w(x,y) = q \cdot C \cdot \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

  where

  a, b = lengths of panel sides

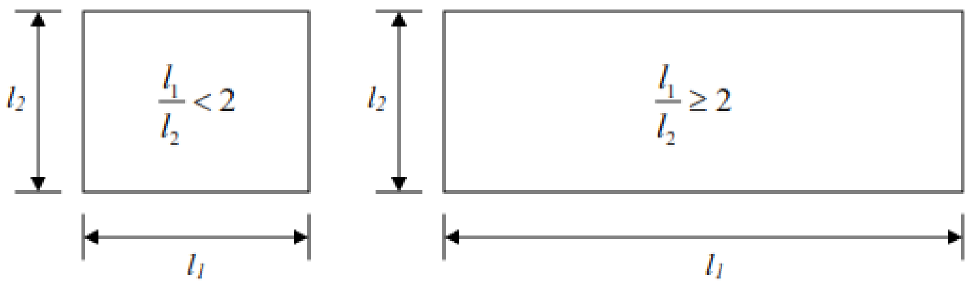
  m, n = integers

  C, A_{mn} = constants – Boundary conditions.

- Finite difference (FD) method
  - It replaces Lagrange's fourth-order PDE with a series of simultaneous linear algebraic equations for the deflections of a finite number of points on the slab surface. Deflections, moments, and shears are computed.
- Finite element (FE) method
  - It utilizes discretization of the physical system into elements. Displacement functions are chosen. Exact compatibility and approximate equilibrium considerations are used.
- Approximate methods
  - Direct design method
  - Equivalent frame method
  - Assignment of moments
- Types of slabs
  - According to the structural action

— One-way slabs

— Two-way slabs



Two-way slab One-way slab

- According to the support and boundary conditions

- Choice of slab type

- Limit analysis – Yield Line Theory
  - Ductility and Yield Line Theory
  - Yield Line Analysis:

    Yield line theory permits prediction of the ultimate load of a slab system by postulating a collapse mechanism which is compatible with the boundary conditions. Slab sections are assumed to be ductile enough to allow plastic rotation to occur at critical section along yield lines.

    1. Postulate a collapse mechanism compatible with the boundary conditions
    2. Moment at plastic hinge lines ≈ Ultimate moment of resistance of the sections
    3. Determine the ultimate load
    4. Redistributions of bending moments are necessary with plastic rotations.

- Moment-curvature relationship

  – Curvature ductility factor: $\frac{\phi_u}{\phi_y}$

  – $\varepsilon_E << \varepsilon_{Plastic}$

  – $M_u \approx$ constant at yield lines.

- Determinate structures → mechanism
- Indeterminate structures – moment redistribution

- Assumptions and guidelines for establishing axes of rotation and yield lines
- Determination of the ultimate load (or moment):
  - Equilibrium method
  - Analysis by Principle of Virtual Work

Isotropic and orthotropic slabs

- Isotropic slabs

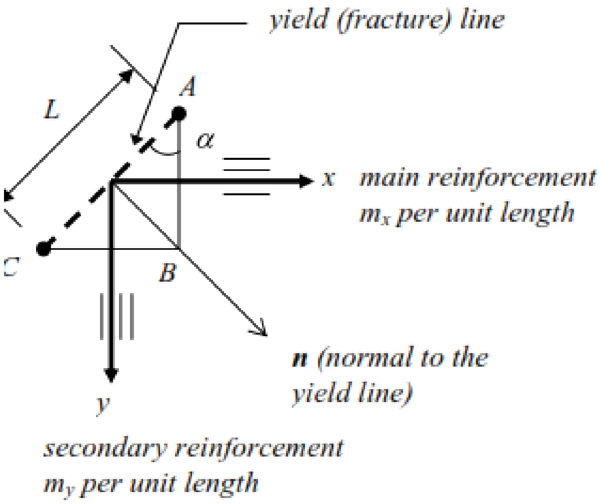
  A slab is said to be isotropically reinforced if it is reinforced identically in orthogonal directions and its ultimate resisting moment is the same in these two directions as it is along any line regardless of its direction.

- Orthotropic slabs

  A slab is said to be orthotropically reinforced if its ultimate strengths are different in two perpendicular directions. In such cases, yield lines will occur across these orthogonal directions.

- Determination of the moment capacity M_u for orthotropic slabs

  Computation for the moment capacity M_u consistent with the yield line given the moment capacities M_x and M_y in the direction of the reinforcing bars:



m_x = *ultimate resisting moment per length along the x axis*

m_y = *ultimate resisting moment per length along the y axis*

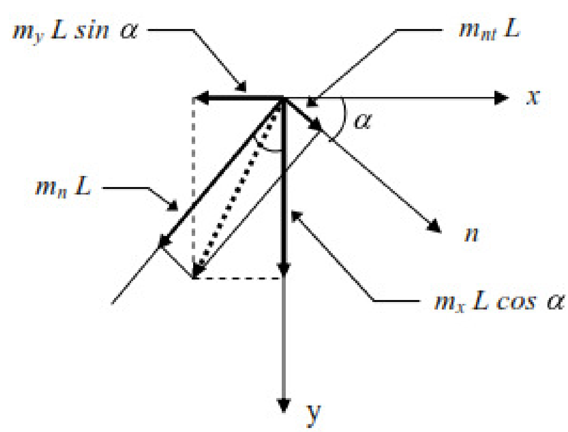
m_n = *ultimate resisting moment per length along AC*

m_{nt} = *ultimate resisting moment per length along normal direction to the yield line (torsion)*

- Equilibrium in vector notation

$m_x\,(AB) = m_x\,L\cos\alpha$
$m_y\,(BC) = m_x\,L\sin\alpha$
$m_n\,(AC) = m_n\,L$
$m_{nt}\,(AC) = m_{nt}\,L$

$m_n\,L = (\,m_x\,L\cos\alpha\,)\cos\alpha + (\,m_y\,L\sin\alpha\,)\sin\alpha$
$m_n = m_x\cos^2\alpha + m_y\sin^2\alpha$
$m_{nt}\,L = (\,m_x\,L\cos\alpha\,)\sin\alpha - (\,m_y\,L\sin\alpha\,)\cos\alpha$
$m_{nt} = (\,m_x - m_y\,)\,L\sin\alpha\cos\alpha$



if $\alpha = 0$ *or* $\pi/2$, *then* $m_{nt} = 0$
if $m_x = m_y = m$, *then*
$m_n = m\,(\cos^2\alpha + \sin^2\alpha)$, $m_{nt} = 0$
$m_n = m$

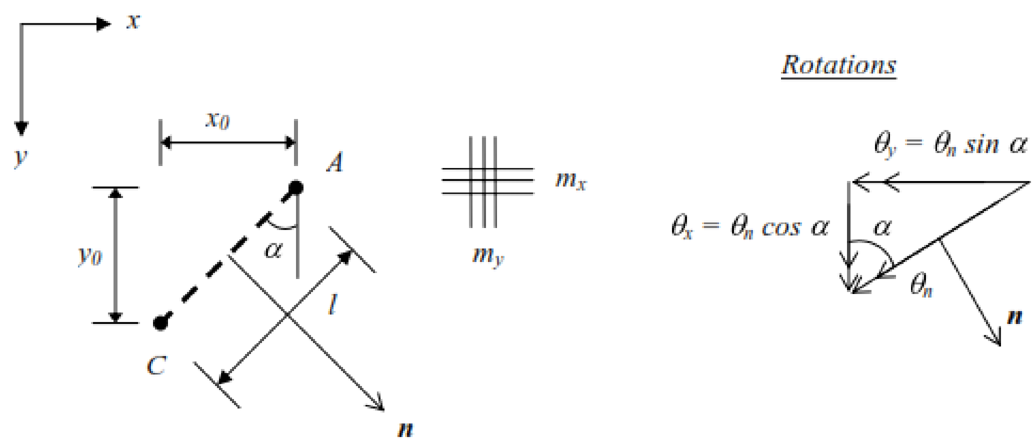
Square yield criterion (isotropic reinforcement)

if $m_x \neq m_y$, *then*
orthogonally anisotropic or orthotropic

Orthotropic slabs can be reduced to equivalent isotropic cases by modifying the slab dimensions.

In analyzing orthotropic plates it is usually easier to deal separately with the x and y direction components of the internal work done by the ultimate moments: $\sum m_n \theta_n l$

- Components of internal work done:



Equilibrium:

$$\sum m_n \theta_n L = \sum \left(m_x \theta_n \cos\alpha \cdot y_0 + m_y \theta_n \sin\alpha \cdot x_0 \right) = \sum \left(m_x \theta_x \cdot y_0 + m_y \theta_y \cdot x_0 \right)$$

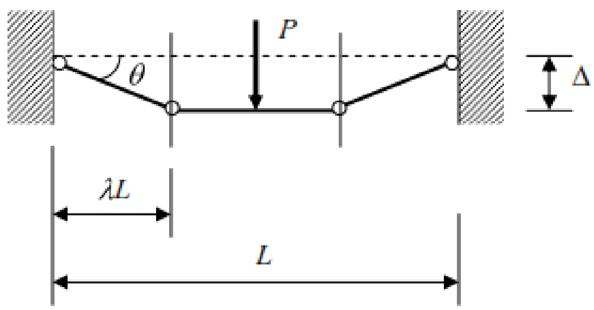
Virtual Work:

$$\sum W \Delta = \sum m_x \theta_x y_0 + \sum m_y \theta_y x_0$$

- Upper-Bound Solution (Energy Approach)
  - Energy method, with an initial selection of a collapse mechanism, gives an upper bound solution, i.e., if failure mode (mechanism) is chosen incorrectly (still satisfying boundary condition) the solution for the ultimate load will be unconservative. The method involves:
    - Select a failure (collapse) mechanism which satisfies the displacement boundary conditions everywhere (kinematic admissibility), and which satisfies the yield criterion at the yield line.

- Impose the condition that work done by the external loads must equal the work done by the resisting forces.
- If the correct mechanism is chosen the method leads to the correct value, otherwise, the predicted load is unconservative.
- This is explained with the following example:

  A fixed ended beam has a positive and negative moment capacity of M_u. Assume the following collapse mechanism,



Conservation of energy:

$$P_u\Delta = 4M_u\theta = 4M_u\frac{\Delta}{\lambda L}$$

$$\Rightarrow P_u = \frac{4M_u}{\lambda L}$$

The correct collapse load is found for $\lambda = 0.5$, $P_u = \frac{8M_u}{L}$. For any other value of $\lambda < 0.5$, P_u is unconservative.

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- Comments on yield line theory:

1. In the equilibrium method, equilibrium of each individual segment of the yield pattern under the action of its bending and torsional moments, shear forces and external loads is considered.
2. In the virtual work method, shear force and torsional moment magnitudes and distribution need not be known because they do not work when summed over the whole slab when the yield line pattern is given a small displacement.

- Limitations on yield line theory:

1. Analysis is based on rotation capacity at the yield line, i.e., lightly reinforced slabs.
2. The theory focuses attention on the moment capacity of the slab. It is assumed an earlier failure would not occur due to shear, bond, etc.
3. The theory does not give any information on stresses, deflections, or service load conditions.

Problem 3: Determine the uniformly distributed collapse load w kN/m^2 of a square slab (L m □ L m), simply supported at three edges and free at the other edge. Assume $M_y = M_x = M$.

Solution 3: The slab is shown in Fig. 12.33.1a. Given data are:

$$L_x = L_y = L \text{ and } M_y = M_x = M.$$

Step 1. **To examine the possibility of yield patterns 1, 2 or both**

From Eq. 12.45 of Lesson 32, it is seen that $(M_y/M_x) < 4\,(L_y/L_n)^2$. So, yield pattern 1 has to be considered. But, (M_y/M_x) is not greater than $(4/3)\,(L_y/L_x)^2$ (vide Eq. 12.54 of Lesson 32). So, yield pattern 2 is not to be considered.

Step 2. **Value of y for yield pattern 1**

Equation 12.43 of Lesson 32 gives: $4y^2 + 2Ly - 3L^2 = 0$ (13)

The solution of Eq.13 is: $y = 0.651\,L$ (14)

Step 3. **Collapse load w kN/m^2**

(i) From Eq.12.41 of Lesson 32: $w = (M/L^2)\{(24)/(3-2y)\} = 14.141(M/L^2)$ (15)

(ii) From Eq. 12.42 of Lesson 32: $w = 6M/y^2 = 14.141\,(M/L^2)$ (15)

(iii) From Eq. 12.48 of Lesson 32: $w = (M/L^2)\,\{(24y + 6)/\{3y-y^2)\} =$

$$14.141(M/L^2) \quad (15)$$

Problem 4: Determine the uniformly distributed collapse load w kN/m^2 of a rectangular slab whose $L_y/L_x = 0.4$, simply supported at three edges and free at the other edge. Assume $M_y/M_x = 1$.

Solution 4: The slab is shown in Fig.12.33.1a. Given data are: $L_y/L_x = 0.4$ and $M_y/M_x = 1$.

Step 1: **To examine the possibility of yield patterns 1, 2 or both**

Here, (M_y/M_x) is not less than 4 $(L_y/L_x)^2$. So, yield pattern 1 is not to be considered. However, (M_y/M_x) is 〉 (4/3) ($L_y/L_x)^2$. So, yield pattern 2 has to be considered (vide Eqs. 12.45 and 12.54 of Lesson 32).

Step 2: **Value of *x* for yield pattern 2**

Equation 12.52 (Lesson 32) gives: $3x^2 + 0.64\,L_x^2 - 0.48\,L_x^2 = 0$ (16)

The solution of Eq. 16 is: $x = 0.3073\,L_x$ (17)

Step 3: **Collapse load *w* kN/m²**

(i) Eq.12.50 gives: $w = (M_x/L_x^2)\,\{6(0.16\text{-}x^2\}/0.16x^2\} = 26.032\,(M_x/L_x^2)$ (18)

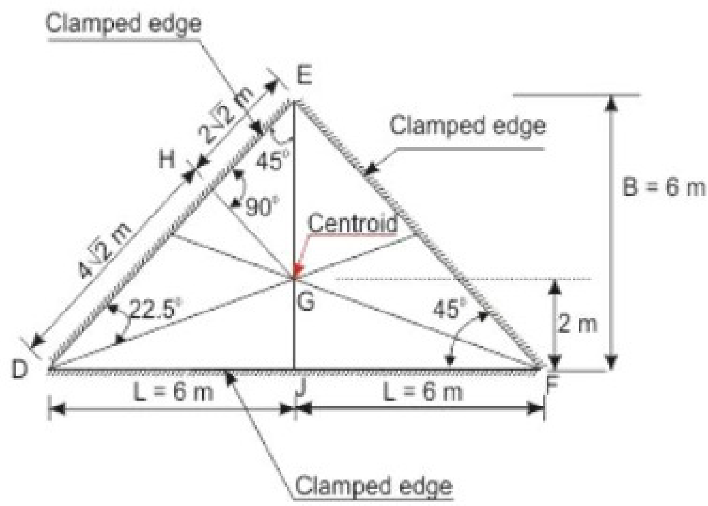
(ii) Eq.12.51 gives: $w = (M_x/L_x^2)\{24x\,/\,0.16(3\text{-}4x)\} = 26.032\,(M_x\,/\,L_x^2)$ (18)

(iii) Eq. 12.57 gives: $w = (M_x/L_x^2)\{12\,(0.16+x^2)/0.16(3x - 2x^2)\}$
$= 26.032\,(M_x/\,L_x^2)$ (18)

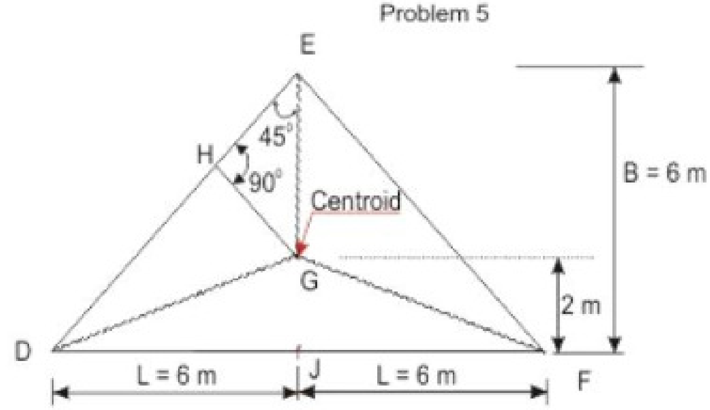
Problem 5: Determine the correct yield pattern and the collapse point load *P* of the isosceles triangular slab of Q.4b of sec.12.31.7 of Lesson 31

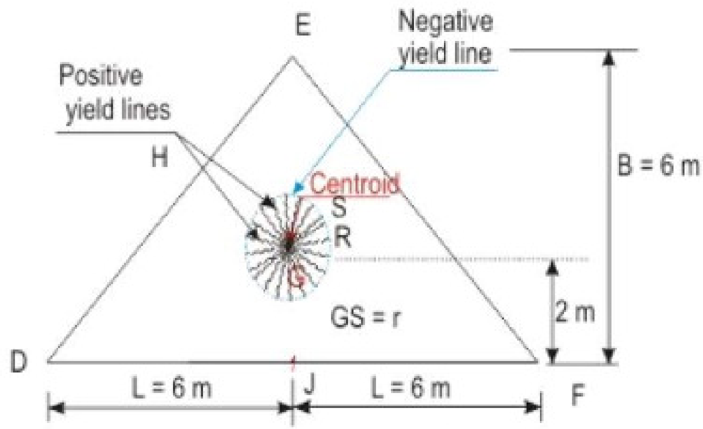
(Fig.12.31.10), if the load is applied at the centroid of the slab. The slab is also shown in Fig. 12.33.2a having B = 6 m and 2*L* = 12 m. Assume M_p = 9 kNm/m and M_n = 12 kNm/m. Use the method of virtual work.

Triangular slabs



Problem 5





Yield pattern 2

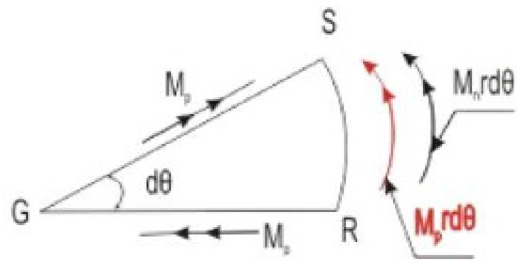


Fig. 12.33.2(d): Free body diagram of segment GSR

Solution 5: The slab is shown in Fig. 12.32.2a. Given data are: 2*L*=12 m, B = 6 m, M_p = 9 kNm/m and M_n = 12 kNm/m. In this problem, two possible yield patterns, as shown in Figs.12.33.2b and c, are to be considered. The lower of the two loads shall be taken, as the correct

collapse load and the corresponding yield pattern is the correct one.

Step 1: **Yield pattern 1**

Yield pattern 1, shown in Fig. 12.33.2b, divides the slab into three segments. Assuming the deflection of the slab at the centroid G = the rotation θ_1 of segment DFG = θ_1 = /2 (as the distance GJ = 2 m). The length of the side DE = 6√2 m and the perpendicular

Distance from G to ED is GH = EG sin 45^0 = 2√2 m. The rotation of the segment DEG = rotation of the segment FEG = /GH = /2√2.

Step 2: ***TIW* and *TEW* due to moments and loads**

TIW = (M_n + M_p) {DF (θ_1) + 2DE (θ_2)} = 21{12(/2) + 2(6√2) (/2√2)} = 252

So, *TIW* = 252 (19)

TIW = *P* (20)

Step 3: **Collapse load *P***

Equating *TIW* and *TEW* from Eqs. 19 and 20, we have, *P* = 252 kN (21)

Thus, the collapse load for the yield pattern 1 is 252 kN.

Step 4: **Yield pattern 2**

Yield pattern 2 divides the segment into a large number of sub-segments as shown in Fig. 12.33.2c. The free body diagram of a typical segment GSR is shown in Fig. 12.33.2d. The internal work done by moments M_p and M_n of the segment GSR is given below.

Internal work done by moments

$$= M_p r\, d\theta\,(\ \ /r) + M_n r\, d\theta\,(\ \ /r) = (M_n + M_p)\, d\theta \qquad (22)$$

Total number of such segment = $2\pi/d\theta$ (23)

So, $TIW = (M_n + M_p)\, d\theta \quad (2\pi/d\theta) = (M_n + M_p)\, 2\pi\,(\ \ )$ (24)

Total external work done by the load:

$$TEW = P \qquad (25)$$

Equating *TIW* and *TEW* from Eqs. 24 and 25, we have

$$(M_n + M_p)\, 2\pi\,(\ \ ) = P$$

which gives

$$P = (M_n + M_p)\, 2\pi = 42\pi = 119.428 \text{ kN} \qquad (26)$$

Step 5: Correct yield pattern and the collapse load

The correct yield pattern is the second one as it gives the lower collapse load P =119.428 kN